A novel approach to the reflection and transmission of electromagnetic waves at interfaces between isotropic and general anisotropic media

Abstract – **An analysis of interfaces between isotropic and general anisotropic media, in a geometric algebra framework, is herein presented. A simple geometric approach, for certain cases of aligned anisotropy, is undertaken and compared to a more powerful, and cumbersome, approach for general anisotropy. The components of the transmitted wavenumbers for an isotropic – uniaxial and an isotropic – indefinite interface are obtained, using respectively the first and the later approach.**

I. INTRODUCTION

The concept of anisotropy is widely explored in electromagnetism, mainly in optics, photonics and, more recently, due to the advent of metamaterials, in the design of cloaking devices [1]. A coordinate free approach has revealed to be the best way to study such media. The classical coordinate-free approach is a dyadic (or tensor) analysis [2]. Some attempts using differential forms have also been presented [3], although both approaches tend to overshadow a direct geometric interpretation of the results as problems escalate. The authors believe that such a geometric interpretation can be achieved by addressing general anisotropy in the geometric framework given by Clifford's algebras [4]-[5]. In fact, new results have been unveiled by using this approach when studying biaxial nonmagnetic crystals [6], thereby unveiling new results that were hidden under the dryness of the dyadic formalism [7]. Further work applying this geometrical framework to reciprocal anisotropic media, where both electric and magnetic anisotropy were considered, led to new results as explained in [8]. The authors have shown that general anisotropic media, with both electric and magnetic functions, must be classified according to the eigenvalues of $\boldsymbol{\zeta}(\mathbf{a}) = \boldsymbol{\varepsilon}^{-1}\left[\boldsymbol{\mu}(\mathbf{a})\right]$: (i) biaxial, when $\zeta_1 < \zeta_2 < \zeta_3$; (ii) uniaxial, when $\zeta_1 = \zeta_2 \neq \zeta_3$; (iii) pseudo-isotropic, when $\zeta_1 = \zeta_2 = \zeta_3$. The only restriction to the previous analysis is that the axes of $\boldsymbol{\varepsilon}(\mathbf{a})$ and $\boldsymbol{\mu}(\mathbf{a})$ should be aligned. This clearly diverges from the classical criteria of classifying a nonmagnetic crystal according to the eigenvalues of $\boldsymbol{\varepsilon}(\mathbf{a})$ (likewise, a magnetic crystal can be classified according to $\boldsymbol{\mu}(\mathbf{a})$) – a criterion that loses its meaning here. Furthermore, none of the eigenwaves of the (general) uniaxial medium is ordinary and neither is the eigenwave of the pseudo-isotropic medium. These newly found results were then applied to isotropic – uniaxial interfaces. By obtaining the eigenwaves for the unbounded uniaxial media, for any given direction, it is possible to split the wavenumbers into two components: (i) $\mathbf{e}_{\parallel}$ parallel to the interface; (ii) $\mathbf{e}_{\perp}$ normal to the interface. The boundary conditions can then be applied in the form of simple algebraic relations, without resorting to the Booker quartic equation. Closed form expressions for the transmitted wavenumbers and for their normal components are then presented as functions of the incidence angle. The conditions for total reflection are also found. Nevertheless, even for certain cases of aligned anisotropy, (e.g. isotropic - biaxial interfaces) the boundary conditions do not take the form of simple algebraic relations. Further work has then been undertaken, where the dispersion equation for the most general case of anisotropic media is derived, considering both electrical and magnetic anisotropy (and assuming that the principal axes of the two anisotropic functions do not need to be aligned). The Booker quartic equation is then obtained and solved for an isotropic – (general) anisotropic interface [10], and applied to an interface between an isotropic and an indefinite medium, defined as $\boldsymbol{\varepsilon}(\mathbf{a}) = \varepsilon\left[\mathbf{a} - 2(\mathbf{a}\cdot\hat{\mathbf{c}})\hat{\mathbf{c}}\right]$ and $\boldsymbol{\mu}(\mathbf{a}) = \mu\mathbf{a}$ in a referential where $\boldsymbol{\varepsilon}(\mathbf{a})$ is anti-diagonal. The transmitted wavenumbers as well as the conditions for total reflection are then obtained. The use of this framework allows for a straightforward geometric interpretation of the electromagnetic behavior of interfaces between air and general anisotropic media which would otherwise be obscured by the cumbersomeness of tensor calculus. Furthermore, it allows to understand whether it is more practical to choose a direct approach, with greater geometrical insight, or to use a more cumbersome yet more powerful approach. Accordingly, in section II.A we solve the interface problem using the more geometrical approach for aligned anisotropy. Section II.B presents the more general approach which can be used for any isotropic – anisotropic interface and is herein applied to an isotropic – indefinite interface. Finally, section III states the conclusions supported by the results of the previous sections.

II. ANALYTICAL APPROACH AND RESULTS

A. Geometric Approach

In $C\ell_3$, the framework of geometric algebra of the three-dimensional Euclidean space [4]-[7], the constitutive relations that describe general anisotropy can be written as $\mathbf{D} = \varepsilon_0\boldsymbol{\varepsilon}(\mathbf{E})$ and $\mathbf{B} = \mu_0\boldsymbol{\mu}(\mathbf{H})$. The linear functions $\boldsymbol{\varepsilon}(\mathbf{a})$ and $\boldsymbol{\mu}(\mathbf{a})$ are the dielectric and magnetic functions that map vectors into vectors, thus defining the angles $\sphericalangle(\mathbf{E},\mathbf{D})$

(electric anisotropy) and $\sphericalangle(\mathbf{H},\mathbf{B})$ (magnetic anisotropy). Furthermore, when considering electromagnetic wave propagation of the form $\exp\left[i k_0 (\mathbf{n}\cdot\mathbf{r} - c t)\right]$, with $\mathbf{n} = n\hat{\mathbf{k}}$ (refractive index), the Maxwell equations can be written in the following form: $\mathbf{n}\wedge\mathbf{E} = c\,\mathbf{B}\,\mathbf{e}_{123}$, $\mathbf{n}\wedge\mathbf{H} = -c\,\mathbf{D}\,\mathbf{e}_{123}$. Then, applying the constitutive relations, it is possible to describe the effect of general anisotropy through the single equation

$$\mathsf{W}(\mathbf{E}_\perp) = \left[\hat{\mathbf{k}}\cdot\mu(\hat{\mathbf{k}})/\det\mu\right] n^2 \zeta(\mathbf{E}_\perp) - \mathbf{E}_\perp, \qquad (1)$$

where

$$\mathbf{E}_\perp = \mathbf{E} - \left\{\mathbf{E}\cdot\left[\mu(\hat{\mathbf{k}})/\hat{\mathbf{k}}\cdot\mu(\hat{\mathbf{k}})\right]\right\}\hat{\mathbf{k}}, \qquad (2)$$

$\zeta(\mathbf{E}_\perp) = \varepsilon^{-1}\left[\mu(\mathbf{E}_\perp)\right]$ and $\mathbf{E}_\perp$ is perpendicular to $\mu(\hat{\mathbf{k}})$. As stated above, (general) anisotropic media will be classified according to the eigenvalues of $\zeta(\mathbf{a})$ defined as $\zeta_i = \mu_i/\varepsilon_i$, where ε_i and μ_i are the eigenvalues of the electric and magnetic functions (respectively). In what follows we will consider a uniaxial medium with $\zeta_1 = \zeta_2 \neq \zeta_3$. From (1), after some algebraic manipulation, it is possible to obtain the dispersion equation. In spherical coordinates, $\hat{\mathbf{k}} = \sin\theta(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + \cos\theta\,\mathbf{e}_3$ ($\mathbf{e}_i$ are the eigenvectors of $\zeta(\mathbf{a})$), the refractive indices of a uniaxial unbounded medium are given by

$$n_+^2 = \frac{\varepsilon_1\varepsilon_2\mu_3\zeta_1}{(\mu_1\cos^2\phi + \mu_2\sin^2\phi)\sin^2\theta + \mu_3\cos^2\theta}, \qquad (3)$$

$$n_-^2 = \frac{\varepsilon_1\varepsilon_2\varepsilon_3\zeta_1}{(\varepsilon_1\cos^2\phi + \varepsilon_2\sin^2\phi)\sin^2\theta + \varepsilon_3\cos^2\theta}, \qquad (4)$$

where the optic axis is $\mathbf{c} = \mathbf{e}_3$, with $n_+^2 = n_-^2 = \varepsilon_1\varepsilon_2\zeta_1$. For a simple anisotropic (nonmagnetic) crystal, (3)-(4) reduce to: $n_+^2 = \varepsilon_\perp\mu$ and $n_-^2 = \varepsilon_\perp\varepsilon_\parallel\mu/(\varepsilon_\perp\sin^2\theta + \varepsilon_\parallel\cos^2\theta)$ (i.e., the refractive indices do not depend on ϕ). Magnetic crystals are similar, although the ordinary wave will then be n_-. To address the problem of an isotropic – uniaxial interface, it is useful to write the wavenumbers as $\mathbf{n}_\alpha = \beta\,\mathbf{e}_\parallel + q_\alpha\mathbf{e}_\perp$, with $\alpha = +,-$, where $\mathbf{e}_\perp = \mathbf{e}_3$ defines the interface plane ($\mathbf{e}_1\wedge\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$) and $\beta\,\mathbf{e}_\parallel = \beta_1\,\mathbf{e}_1 + \beta_2\,\mathbf{e}_2$. After imposing the boundary conditions, $\beta = n_i\sin\theta_i = n_\pm\sin\theta_\pm$ (where n_i is the refractive index of the incident wave in the isotropic medium) we obtain the components of the transmitted waves in the uniaxial medium, as functions of the incidence angle θ_i:

$$q_+ = \sqrt{\left[n_i^2\sin^2\theta_i\left(-\mu_1\cos^2\phi - \mu_2\sin^2\phi\right) + \varepsilon_1\varepsilon_2\mu_3\zeta_1\right]/\mu_3}, \qquad (5)$$

$$q_- = \sqrt{\left[-n_i^2\sin^2\theta_i\left(\varepsilon_1\cos^2\phi + \varepsilon_2\sin^2\phi\right) + \varepsilon_1\varepsilon_2\varepsilon_3\zeta_1\right]/\varepsilon_3}. \qquad (6)$$

Again, for a simple uniaxial (nonmagnetic) crystal, one has $q_+ = \sqrt{\varepsilon_\perp\mu - n_i^2\sin^2\theta_i}$ and $q_- = \sqrt{(\varepsilon_\parallel\mu - n_i^2\sin^2\theta_i)\varepsilon_\perp/\varepsilon_\parallel}$. The following results were obtained adopting $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, $\varepsilon_3 = 3$, $\zeta_1 = \zeta_2 = 4/3$ and $\zeta_3 = 5/2$ for the uniaxial medium and $n_i = 3$ for the isotropic medium. As shown in Fig. 1a, in spite of the fact that both electric and magnetic functions are biaxial, the medium behaves as a uniaxial medium presenting only one optic axis. There is another important aspect worth mentioning: none of the eigenwaves is an ordinary wave. Hence, unlike for the nonmagnetic crystal, it is impossible to adjust the refractive index of the isotropic medium so that $\theta_i = \theta_t$ at the interface. Furthermore, for total reflection, the corresponding conditions $\mathbf{q}_\pm = iq_\pm\mathbf{e}_\perp$ for each eigenwave are given by

$$q_+ \;\mapsto\; n_i^2\sin^2\theta_i\left(\mu_1\cos^2\phi + \mu_2\sin^2\phi\right) > \varepsilon_2\mu_3\mu_1, \qquad (7)$$

$$q_- \;\to\; n_i^2\sin^2\theta_i\left(\varepsilon_1\cos^2\phi + \varepsilon_2\sin^2\phi\right) > \varepsilon_2\varepsilon_3\mu_1, \qquad (8)$$

whereas, for a nonmagnetic crystal, we simply have $q_\pm \;\mapsto\; n_i^2\sin^2\theta_i > \varepsilon_{\perp/\parallel}\mu$. In Fig. 1c, whenever total reflection occurs, q_- is depicted by a dashed line. In Fig. 2 we present the components of the transmitted waves considering a variation with θ_i and ϕ.

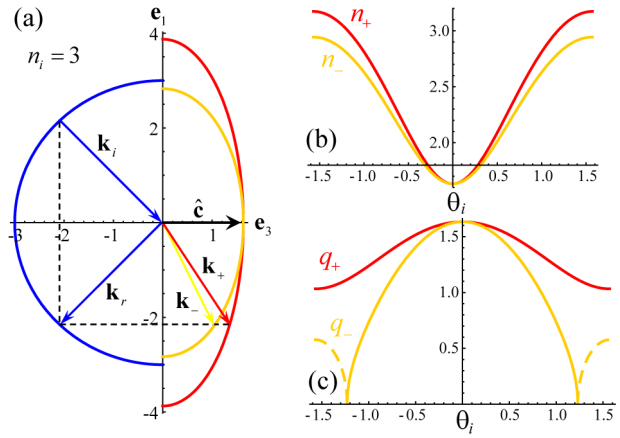
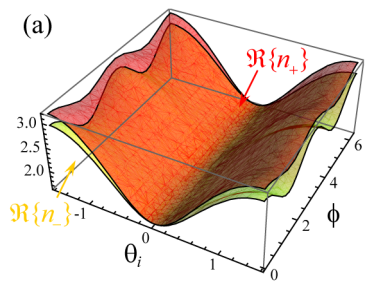


Fig. 1. General uniaxial medium: (a) eigenwaves at $\mathbf{e}_1\wedge\mathbf{e}_3$ and optic axis $\hat{\mathbf{c}}$; (b) $n_\pm$ and (c) $q_\pm$ of the transmitted waves.



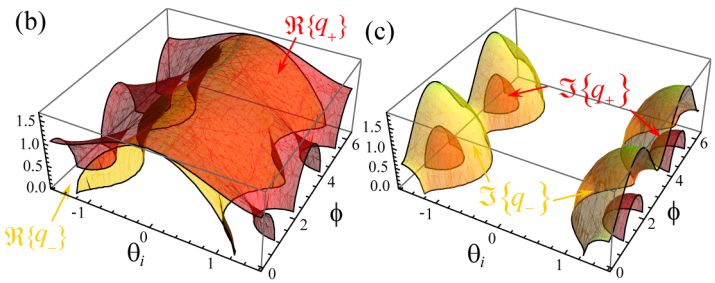


Fig. 2. Transmitted waves: (a) $n_\pm$; (b) $\Re\{q_\pm\}$; (c) $\Im\{q_\pm\}$.

B. General approach

Applying the constitutive relations to the Maxwell equation, we can arrive at a single geometric equation

$$\varepsilon(\mathbf{E})+\left[\mathbf{n}\lrcorner\,\bar{\mu}(\mathbf{E}\wedge\mathbf{n})\right]/\det(\mu)=0\,, \tag{9}$$

from where, considering $\mathbf{n}=n\hat{\mathbf{k}}$, the dispersion equation can be derived

$$an^4+bn^2+c=0\,, \tag{10}$$

where

$$a=\left[\mu(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\right]\left[\varepsilon(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\right],\quad c=\det(\varepsilon)\det(\mu) \tag{11}$$

$$b=\mu(\hat{\mathbf{k}})\cdot\varepsilon^\dagger\left[\mu(\hat{\mathbf{k}})\right]-\left[\mu(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\right]\mathrm{tr}\left[\varepsilon^\dagger(\mu)\right]. \tag{12}$$

Furthermore, we will again write the wavenumbers at an isotropic – (general) anisotropic interface as $\mathbf{n}_\alpha=\beta\mathbf{e}_\parallel+q_\alpha\mathbf{e}_\perp$. Replacing them into (10), and after some algebra, it is possible to arrive at the Booker quartic equation

$$aq_\alpha^4+bq_\alpha^3+cq_\alpha^2+dq_\alpha+e=0\,, \tag{13}$$

where

$$a=\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right], \tag{14}$$

$$\begin{aligned}b=&\,2\beta\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\\&+2\beta\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right],\end{aligned} \tag{15}$$

$$\begin{aligned}c=&\,\beta^2\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right]\left[\varepsilon(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]\\&+4\beta^2\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\\&+\beta^2\left[\mu(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right]+\mu(\mathbf{e}_\perp)\cdot\varepsilon^\dagger\left[\mu(\mathbf{e}_\perp)\right],\\&-\mathrm{tr}\left[\varepsilon^\dagger(\mu)\right]\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\perp\right]\end{aligned} \tag{16}$$

$$\begin{aligned}d=&\,2\beta^3\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]\\&+2\beta^3\left[\mu(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]+\beta\mu(\mathbf{e}_\perp)\cdot\varepsilon^\dagger\left[\mu(\mathbf{e}_\parallel)\right],\\&+\beta\mu(\mathbf{e}_\parallel)\cdot\varepsilon^\dagger\left[\mu(\mathbf{e}_\perp)\right]-2\beta\,\mathrm{tr}\left[\varepsilon^\dagger(\mu)\right]\left[\mu(\mathbf{e}_\perp)\cdot\mathbf{e}_\parallel\right]\end{aligned} \tag{17}$$

$$\begin{aligned}e=&\,\beta^4\left[\mu(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]\left[\varepsilon(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]+\beta^2\mu(\mathbf{e}_\parallel)\cdot\varepsilon^\dagger\left[\mu(\mathbf{e}_\parallel)\right]\\&-\beta^2\,\mathrm{tr}\left[\varepsilon^\dagger(\mu)\right]\left[\mu(\mathbf{e}_\parallel)\cdot\mathbf{e}_\parallel\right]+\det(\varepsilon)\det(\mu)\end{aligned}\,. \tag{18}$$

Now, considering an indefinite medium, defined as $\varepsilon(\mathbf{a})=\varepsilon\left[\mathbf{a}-2(\mathbf{a}\cdot\hat{\mathbf{c}})\hat{\mathbf{c}}\right]$ and $\mu(\mathbf{a})=\mu\mathbf{a}$, but rotated to $\hat{\mathbf{c}}=(\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_3)/\sqrt{2}$, where $\varepsilon(\mathbf{a})$ is anti diagonal. The wavenumbers for the unbounded medium can be obtained from (10),

$$n_+=\sqrt{\varepsilon\mu}\,,\qquad n_-=\sqrt{\frac{\varepsilon\mu}{-1+(k_1-k_3)^2}}\,, \tag{19}$$

where $\mathbf{k}=\{k_1,k_2,k_3\}$. Furthermore, aligning the interface such that $\mathbf{e}_\perp=\mathbf{e}_3$ and $\beta\mathbf{e}_\parallel=\beta_1\mathbf{e}_1+\beta_2\mathbf{e}_2$ we can obtain the transmitted and reflected wavenumbers by solving (13)

$$q_+=\pm\sqrt{-\beta^2+\varepsilon\mu}\,,\qquad q_-=-\frac{\beta_2^2+\varepsilon\mu}{2\beta_1}\,, \tag{20}$$

from where the conditions for total reflection, $\beta^2>\varepsilon\mu$, can be immediately derived. The following results were obtained adopting $n_i=2$ for the isotropic medium, and $\varepsilon=2$, $\mu=2$ for the indefinite medium. In Fig. 3a we can see the incident, reflected and transmitted waves at the interface between both media at plane $\mathbf{e}_1\wedge\mathbf{e}_3$, where the imaginary part of n_- is depicted by a dashed line. In Figs. 3b – 3c the components of the transmitted wavenumbers are shown. Furthermore, the wavenumbers of an unbounded indefinite medium (Fig. 4) and the components of the transmitted wavenumbers (Fig. 5) are shown as functions of θ and ϕ.

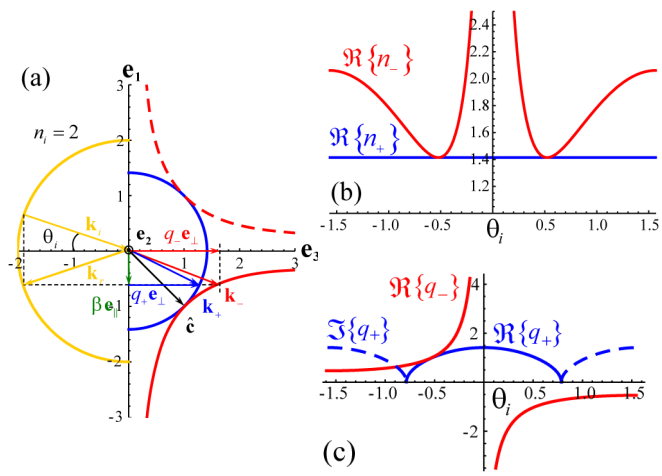


Fig. 3. (a) Eigenwaves at $\mathbf{e}_1\wedge\mathbf{e}_3$ and optic axis $\hat{\mathbf{c}}$; (b) $n_\pm$ and (c) $q_\pm$ of the transmitted waves.

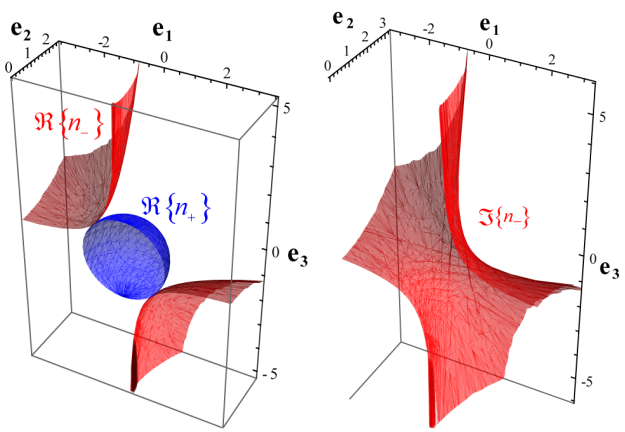


Fig. 4. Eigenwaves on the unbound medium under study: (a) $\Re\{n_\pm\}$; (b) $\Im\{n_\pm\}$.

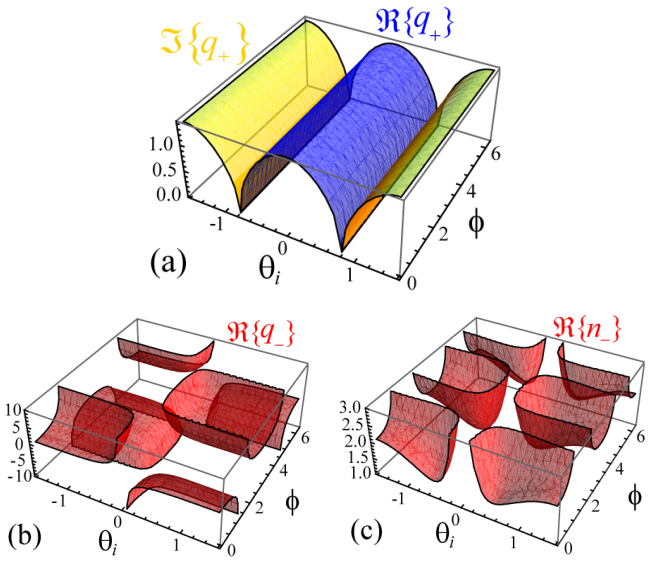


Fig. 5. Transmitted waves: (a) q_+ ; (b) q_- ; (c) n_- .

III. CONCLUSIONS

The problem of interfaces between isotropic and general anisotropic media has been addressed using a geometric algebra framework. In fact, only in this framework is it possible to classify the behavior of general anisotropic media in a clear and insightful way. A more geometric approach has been used to analyze interfaces between isotropic media and media characterized by both biaxial electric and magnetic functions, globally behaving as uniaxial media. The transmitted wavenumber components have been explicitly obtained revealing that, unlike those of simple uniaxial crystals, they exhibit an explicit dependence with the azimuth angle. The conditions for total reflection were also found to depend on this azimuth angle. Additionally, since all the eigenwaves of general uniaxial media are extraordinary, it is not possible to obtain a transmitted angle that is equal to the incident angle for any of the wavenumbers. This can be easily achieved at an isotropic – (simple) uniaxial crystal interface by choosing the refractive index of the isotropic media to equal that of the ordinary wave of the (nonmagnetic) uniaxial crystal. Nevertheless, for certain cases of aligned anisotropy, (e.g. isotropic - biaxial interfaces) the boundary conditions do not take the form of simple algebraic relations. A more general (and cumbersome) approach has then been developed to address the limitation of the previous approach. It allows to solve the problem of interfaces between isotropic and general anisotropic media, where the axis of the constitutive functions do not need to be aligned. The Booker quartic equation can then be obtained and solved for this general case. This approach was herein applied to an interface between and isotropic and an indefinite medium, in a referential where the electric function is anti-diagonal. The transmitted wavenumbers were then obtained, as well as the conditions for total reflection.

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